5.4 Logarithmic Functions

Question: What is the inverse of an exponential function? How do you solve for a variable that is in an exponent?

Find the inverse of $f(x) = 2^x$.

1. Replace $f(x)$ with *y*. $y = 2^x$
2. Interchange *x* and *y*. $x = 2^y$
3. Solve for *y*. $y =$ the exponent to which we raise 2 to get *x*.
4. Replace *y* with $f^{-1}(x)$ $f^{-1}(x) =$ the exponent to which we raise 2 to get *x*.

We need a new symbol to replace the words: "The exponent to which we raise 2 to get *x*":

$\log_2 x$ means "the exponent to which we raise 2 to get *x*."

Pronounced "the logarithm, base 2, of *x*" or "log, base 2, of *x*"

★LOGARITHMS ARE EXPONENTS!★

Logarithm: $\mathbf{\log_b a}$ means the ***exponent*** to which we raise $\boldsymbol{b}$ to get $\boldsymbol{a}$**.**

- $\boldsymbol{b}$ is called the ***base*** of the logarithm (the number being raised to the exponent).
- $\boldsymbol{a}$ is called the ***argument*** of the logarithm (the number you get when you raise the base to the exponent).

The ***logarithmic function of base a***, where $a > 0$ and $a \neq 1$ is denoted by $y = \log_a x$ and is defined by

$$y = \log_a x \text{ if and only if } x = a^y.$$

Example: Change each exponential expression to an equivalent expression involving a logarithm.

a) $5^x = 625$ b) $x^3 = 64$ c) $3^2 = x$

Example: Change each logarithmic expression to an equivalent expression involving an exponent.

a) $\log_3 x = 5$ b) $\log_e 5 = x$ c) $\log_m 2 = n$

Evaluating Logarithms: It is helpful to replace "log" with the word "power".

- Instead of "$\log_2 8$," think "$\text{power}_2\, 8$." Ask yourself, what power of 2 equals 8?
  - The answer would be 3 because $2^3 = 8$.

Example: Find the exact value of

a) $\log_3 9$ b) $\log_2 32$ c) $\log_6 1$ d) $\log_5 \frac{1}{125}$ e) $\log_7 \sqrt{7}$

Domain of a Logarithmic Function

The logarithmic function $y = \log_a x$ is the inverse of the exponential function $y = a^x$.

Domain of the logarithmic function = Range of the exponential function $= (0, \infty)$

Range of the logarithmic function = Domain of the exponential function $= (-\infty, \infty)$

$y = \log_a x$ (defining equation: $x = a^y$)

Domain: $(0, \infty)$ Range: all real numbers

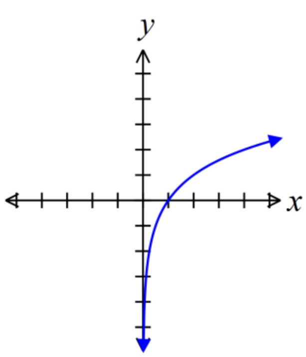
★ **Caution!** You can't take the log of zero or of a negative because it is impossible to get zero or a negative by raising a positive base to an exponent. **The argument of a logarithmic function must be greater than zero.**

Example: Find the domain of each logarithmic function

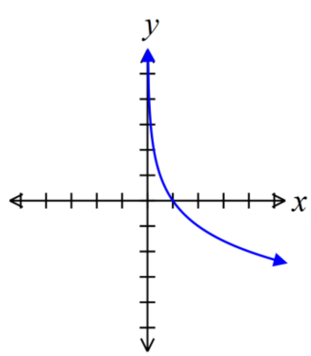
a) $f(x) = \log_2(x+3)$ b) $g(x) = \log_5\left(\dfrac{1+x}{1-x}\right)$ c) $h(x) = \log_{\frac{1}{2}}|x|$

Graphs of Logarithmic Functions

$f(x) = \log_a x,\ a > 1$



$f(x) = \log_a x,\ 0 < a < 1$



Properties of the Logarithmic Function $f(x) = \log_a x$

1. The domain is the set of all positive real numbers; the range is the set of all real numbers.
2. The x-intercept is 1. There is no y-intercept.
3. The y-axis $(x = 0)$ is a vertical asymptote of the graph.
4. The logarithmic function is decreasing if $0 < a < 1$ and increasing if $a > 1$. The function is one-to-one.
5. The graph of f contains the points $(1, 0)$, $(a, 1)$, and $\left(\dfrac{1}{a}, -1\right)$.
6. The graph of f is smooth and continuous, with no corners, gaps, or cusps.

★ **Note:** It is often easier to graph a logarithmic function if you rewrite it as an exponential function first.

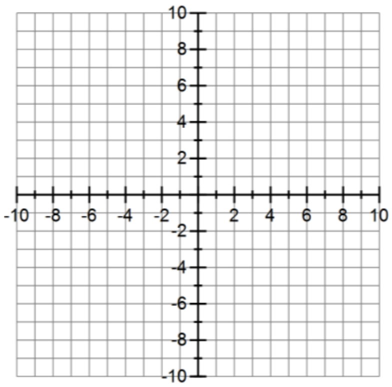
Graphing Logarithmic Functions:

1. Solve the equation for x by rewriting it as an exponential function.

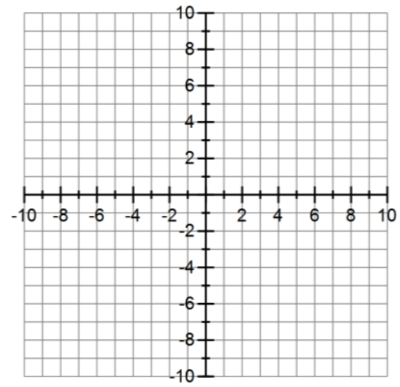
2. Choose y-values, and plug them in to find the x-values.
3. Plot your points and connect them to form a smooth curve.

Examples:

a) Graph $y = 2^x$ and $y = \log_2 x$



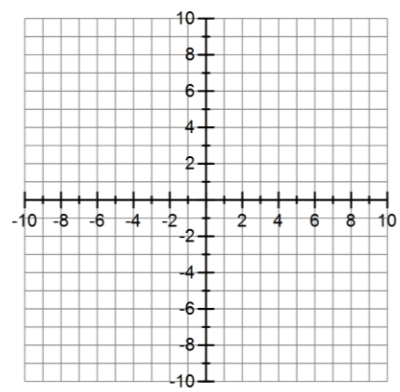
b) Graph $y = \left(\frac{1}{3}\right)^x$ and $y = \log_{1/3} x$.



Natural Logarithms: If the base of a logarithmic function is the number e, then we have the natural logarithm function (abbreviated ln). That is, $y = \ln x$ if and only if $x = e^y$.

Example: $f(x) = -\ln(x+3)$

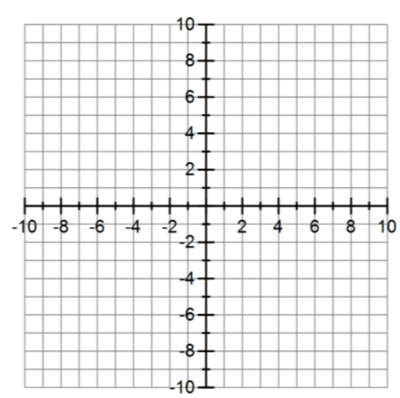
a) Find the domain of the logarithmic function.
b) Graph $f(x)$.
c) Find the range and vertical asymptote of f.
d) Find f^{-1}, the inverse of f.
e) Graph f^{-1}.



Common Logarithmic Function: If the base of a logarithmic function is the number 10, then we have the common logarithmic function. If the base a of the logarithmic function is not indicated, it is understood to be 10. That is, $y = \log x$ if and only if $x = 10^y$.

Example: $f(x) = 2\log(x-3)$

a) Find the domain of the logarithmic function.
b) Graph $f(x)$.
c) Find the range and vertical asymptote of f.
d) Find f^{-1}, the inverse of f.
e) Graph f^{-1}.



Solving Logarithmic Equations

Many equations can be solved by rewriting logarithms as exponential functions or rewriting exponential functions as logarithms.

★ When solving logarithmic equations, remember that in the expression $\log_a M$, a and M must be positive and $a \neq 1$. Be sure to check each solution in the original equation and discard any that are extraneous.

Examples: Solve the logarithmic equations

a) $\log_3(3x-2)=2$

b) $\log_x\left(\frac{1}{8}\right)=3$

c) $10^{2x-7}=3$

d) $e^{3x-2}=7$

e) $\log_2(x^2+2x)=3$

f) $4e^{x+1}=5$

Example: The blood alcohol concentration (BAC) is the concentration of alcohol in a person's bloodstream. The relative risk of having an accident while driving a car is given by the equation $R=e^{kx}$, where R is the relative risk (how many times more likely a person with a certain BAC is to have a car accident than a person who has not been drinking), x is the BAC (expressed as a percentage), and k is a constant.

a) If the relative risk is 1.4 when the blood concentration is 0.02%, find k.

b) Using k from part a), find the relative risk if the blood alcohol concentration is 0.17%.

c) What BAC corresponds to a relative risk of 100?